Stochastic Processes Recap

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1 Fundation of probability theory

Definition. *Let Ω be a set. A σ algebra of Ω is a set B $\subset \mathcal{P}(\Omega)$ such that:*
(1) $\emptyset,\ \Omega \in B$
(2) $A \in B \Longrightarrow A^c \in B$
(3) $A_1, A_2, \cdots \in B \Longrightarrow \bigcup_{i=1}^{\infty} A_i \in B$

Definition. *If A is a family of subsets of Ω, then one can define the σ algebra generated by A to be the intersection of all σ algebras containing A.*

Definition. *If X is a topological space then the Borel σ algebra of X is the σ generated by the open sets.*

Definition. *A probability Space is a triple $(\Omega,\ \mathcal{F},\ \mathbb{P})$, where*
(1) $\Omega \neq \emptyset$
(2) $\mathcal{F}$ *is a σ algebra on Ω.*
(3) $\mathbb{P} : \mathcal{F} \longrightarrow [0, 1]$ *satisfies*
(i) $\mathbb{P}(\Omega) = 1$
(ii) $\mathbb{P}(A^C) = 1 - \mathbb{P}(A)$
(iii) $\{A_i\}$ *with* $A_i \cap A_j = \emptyset, \mathbb{P}(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mathbb{P}(A_j)$

Definition. *An element of $\mathcal{F}$ is called an event.*

Proposition.
(1) $A \subseteq B, A, B \in \mathcal{F}$*, then* $\mathbb{P}(A) \leq \mathbb{P}(B)$.
(2) If $A_1 \subseteq A_2 \subseteq A_3 \subseteq \ldots$ *in the probability space* $(\Omega,\ \mathcal{F},\ \mathbb{P})$*. Then* $\mathbb{P}(\bigcup_{i=1}^{\infty} A_i) = \lim_{i\to\infty} \mathbb{P}(A_i)$
(3) If $A_1 \supseteq A_2 \supseteq A_3 \supseteq \ldots \mathbb{P}(\bigcap_{i=1}^{\infty} A_i)$ = $\lim_{i\to\infty} \mathbb{P}(A_i)$

Definition. *Let $A_1, A_2, \ldots$ be a sequence of sets,*

$$\limsup_{n\to\infty} A_n = \{x : \text{ there are infinitely many } j \text{ such that } x \in A_j\} = \bigcap_{i=1}^{\infty} \bigcup_{n=i}^{\infty} A_n$$

$$\liminf_{n\to\infty} A_n = \{x : \text{ for all but finitely many } j \text{ such that } x \in A_j\} = \bigcup_{i=1}^{\infty} \bigcap_{n=i}^{\infty} A_n$$

Definition. *$(\Omega,\ \mathcal{F},\ \mathbb{P})$ is a probability space. $A, B \in \mathcal{F}$.*
We say A and B are independent if $\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$.
The conditional probability A occurs given B occurs is: $\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$

Definition. *Let $\{A_i\}_{i\in I}$ be a family of events. We call this family independent if for any of distinct indices $i_1, \ldots, i_k$,*

$$\mathbb{P}(A_{i_1} \cap A_{i_2} \cap \ldots A_{i_k}) = \mathbb{P}(A_{i_1})\mathbb{P}(A_{i_2}) \ldots \mathbb{P}(A_{i_k})$$

Definition. *Assume $\{X_i\}_{i\in I}$ is a form of random variables defined on $(\Omega,\ \mathcal{F},\ \mathbb{P})$. We call $\{X_i\}_{i\in I}$ indepedent if $\forall \{A_i\}_{i\in I}$, events $X_i^{-1}(A_i) = \{w : X_i(w) \in A_i\}$ are independent.*
X, Y are independent if $\forall A, B \in \mathcal{B}$

$$\mathbb{P}(X^{-1}(A) \cap Y^{-1}(B)) = \mathbb{P}(X^{-1}(A))\mathbb{P}(Y^{-1}(B)), \quad \mathbb{P}_{xy} = \mathbb{P}_x\mathbb{P}_y$$

Definition. *$X_1, \ldots, X_n$ are random variables. The joint distribution function of $X_1, \ldots, X_n$ is*

$$F_X(x_1, \ldots, x_n) = \mathbb{P}(x_1 \leq X_1, \ldots, x_n \leq X_n)$$

Particular, if $x_1, \ldots, x_n$ are independent,

$$F_X(x_1, \ldots, x_n) = \Pi_{i=1}^n F_X(x_i)$$

Definition. *Let $(\Omega,\ \mathcal{F},\ \mathbb{P})$ be a probability space. Let $X : \Omega \longrightarrow \mathbb{R}$ be a function. X is called a random variable if for any Borel set $\mathcal{B} \in \mathbb{R}$, $X^{-1}(\mathcal{B}) \in \mathcal{F}$.*

Definition. *Let X be a random variable in $(\Omega,\ \mathcal{F},\ \mathbb{P})$ as above, Define the probability measure on $\mathbb{R}$ as follows:*
Let $A \in \mathcal{B}_{\mathbb{R}}$. $\mathbb{P}_X(A) = \mathbb{P}(X^{-1}(A)) = \mathbb{P}(X \in A)$
$\mathbb{P}_X$ is called the distribution of X.
$\mathbb{P}_X(\mathbb{R}) = \mathbb{P}(X \in \mathbb{R}) = 1$

Definition. *$F_X(x) = \mathbb{P}(X^{-1}(-\infty, x]) = \mathbb{P}(X \leq x)$ $F_X : \mathbb{R} \longrightarrow [0, 1]$, is called the cumulative distribution function of X.*
$F_X(x) = \mathbb{P}_X((-\infty, x])$

Proposition. *A cumulative distribution function F has the following properties:*
(1) $\lim_{x\to-\infty} F(x) = 0$
(2) if $x < y$ then $F(x) < F(y)$
(3) F is right-continuous, that is, $F(x + h) \to F(x)$ as $h \downarrow 0$ (4) $\mathbb{P}(x > t) = 1 - F(t)$
(5) $\mathbb{P}(p < x \leq q) = F(q) - F(p)$
(6) $\mathbb{P}(x = t) = F(t) - \lim_{y\uparrow t} F(y)$

2 Discrete Random Variables

Definition. *The random variable X is called discrete if it takes values in some countable subset $\{x_1, x_2, \ldots\}$ only of $\mathbb{R}$. The discrete random variable X has **probability mass function**. $f : \mathbb{R} \longrightarrow [0, 1]$ given by $f(x) = \mathbb{P}(X = x)$*

Examples of Discrete Random Variables

Distribution	Probability Mass Function
Bernoulli	$\mathbb{P}(X = 1) = p, \mathbb{P}(x = 0) = 1 - p$
Binomial	$\mathbb{P}(X = k) = \binom{n}{k} p^k(1-p)^{n-k}$
Geometric	$\mathbb{P}(X = k) = p(1-p)^{k-1}$
Possion	$\mathbb{P}(X = k) = e^{-\lambda}\frac{\lambda^k}{k!}$

Definition. *The discrete random variable X and Y are independent if event* $\{X = x\}$ *and* $\{Y = y\}$ *are independent for all x and y.*

Theorem. *Discrete random variable* X *and* Y *are independent random variable.* g*,* h *are function map from* $\mathbb{R}$ *to* $\mathbb{R}$*. Then* $g(X)$ *and* $h(Y)$ *are independent also.*

Definition. *The expectation of discrete random variable X with mass function f is defined to be:*

$$\mathbb{E}(X) = \sum_{x, f(x)>0} x f(x) = \sum x \mathbb{P}(X = x)$$

Lemma. *If X has mass function* f*.* $g : \mathbb{R} \longrightarrow \mathbb{R}$*. Then*

$$\mathbb{E}(g(X)) = \sum_x g(x) f(x) = \sum_x g(x) \mathbb{P}(X = x)$$

Theorem. *Let X be a random variable.*
(1) $X \geq 0, \mathbb{E}(x) \geq 0$
(2) $a, b \in \mathbb{R}, \mathbb{E}(aX + bY) = a\mathbb{E}(X) + b\mathbb{Y}$

Definition. X *and* Y *are called uncorrelated if* $\mathbb{E}(XY) = \mathbb{E}(X)\mathbb{E}(Y)$

Lemma. *If* X *and* Y *are independent random variable. Then*

$$\mathbb{E}(XY) = \mathbb{E}(X)\mathbb{E}(Y)$$

Definition.

$$\textit{Var}X = \mathbb{E}\left[(X - \mathbb{E}(X))^2\right] = \mathbb{E}(X^2) - \mathbb{E}(X)^2$$

Theorem.

$$(\mathbb{E}(XY))^2 \leq \mathbb{E}(X^2)\mathbb{E}(Y^2)$$

3 Continuous Random Variable

Definition. *The random variable X is called continuous if its distribution function can be expressed as*

$$F(x) = \int_{-\infty}^{x} f(u) du, x \in \mathbb{R}$$

for some integrable function $f : \mathbb{R} \longrightarrow [0, \infty)$ *called* ***probability density function****.*

Examples of Continuous Random Variables

Distribution	Probability Mass Function	Probability Density Function
Uniform	$f(x) = \begin{cases} \frac{1}{b-a} & : x \in [a, b] \\ 0 & : else \end{cases}$	$F(x) = \begin{cases} 0 & x \leq a \\ \frac{x-a}{b-a} & x \in [a, b) \\ 1 & x \leq 1 \end{cases}$
Exponential	$f(x) = \begin{cases} \lambda e^{-\lambda x} & : x \geq 0 \\ 0 & : x \leq 0 \end{cases}$	$F(x) = \begin{cases} 1 - \lambda e^{-\lambda x} & : x \geq 0 \\ 0 & : x \leq 0 \end{cases}$
Normal	$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}(\frac{x-\mu}{\sigma})^2}$	$\frac{1}{2}\left[1 + \text{erf}(\frac{x-\mu}{\lambda\sqrt{2}})\right]$

Lemma. X *has density function f.*
(a) $\int_{-\infty}^{\infty} f(x) dx = 1$
(b) $\mathbb{P}(X = x) = 0, \forall x \in \mathbb{R}$
(c) $\mathbb{P}(a \leq X \leq b) = \int_a^b f(x) dx$

Definition. *Continuous random variable X and Y are called independent if $\{X \leq x\}$ and $\{Y \leq y\}$ are independent events, $\forall x, y \in R$.*

Theorem. *Continuous random variable X and Y are independent random variable. g, h are function map from $\mathbb{R}$ to $\mathbb{R}$. Then $g(X)$ and $h(Y)$ are independent also.*

Definition. *The expectation of a continuous random variable X with density function f is given by*

$$\mathbb{E}(x) = \int_{-\infty}^{\infty} xf(x)dx$$

Theorem. *If X, g(X are continuous random variable. Then*

$$\mathbb{E}(g(x)) = \int_{-\infty}^{\infty} g(x)f_X(x)dx$$

Lemma. *If X has density function f, $f(x) = 0$, $x < 0$ and distribution function $F(x)$, then*

$$\mathbb{E}(X) = \int_{0}^{\infty} 1 - F(x)dx$$

4 Expectation Revisit

Recall

Definition. *The expectation of discrete random variable X with mass function f is defined to be:*

$$\mathbb{E}(X) = \sum_{x, f(x)>0} xf(x) = \sum x\mathbb{P}(X = x)$$

The expectation of a continuous random variable X with density function f is given by

$$\mathbb{E}(X) = \int_{-\infty}^{\infty} xf(x)dx = \int_{-\infty}^{\infty} x\mathbb{P}(X = x)dx$$

Theorem. *A random variable X defined on $(\Omega, \mathcal{F}, \mathbb{P})$ is called simple if it takes finitely many distinct values.*

$$X = \sum_{i=1}^{n} x_i I_{A_i} \textit{ for some } A_1, \ldots, A_n \textit{ of } \Omega \textit{ and some real numbers } x_1, \ldots, x_n$$

Definition. *Any non negative random variable $X : \Omega \longrightarrow [0, -\infty)$ is the limit of some increasing sequence $\{X_n\}$ of simple variables $X_n(w) \uparrow X(w)$. Let $x \geq 0$ be arbitrary random variable. $\mathbb{E}(X) := sup\{\mathbb{E}(Y) : 0 \leq Y \leq X, Y simple\}$*

Theorem. *Let $X \geq 0$ be an arbitrary random variable. $X \geq 0$. Let X_n, $Y > 0$ be also random variable with $X_n \uparrow X$ and $0 \leq Y \leq X$. X_n is simple. Then $\lim_{n\to\infty} \mathbb{E}(X_n) \geq \mathbb{E}(Y)$.*

Corollary. *Let $X \geq 0$, $X_n \uparrow X$ be a sequence of non-negative simple random variable. $\mathbb{E}(X) = \lim_{n\to\infty} \mathbb{E}(X_n)$*

Theorem. *$\phi : [a, b] \longrightarrow \mathbb{R}$ is called convex if $\forall x_1, x_2 \in [a, b]$, $\forall \lambda \in [0, 1]$ $\phi(\lambda x_1 + (1 - \lambda)x_2) \leq \lambda\phi(x_1) + (1 - \lambda)\phi(x_2)$*

Theorem. *If X is an integrable random variable. $\phi : \mathbb{R} \longrightarrow \mathbb{R}$ is a convex function $\mathbb{E}(\phi(x)) \geq \phi(\mathbb{E}(X))$*

Theorem. *If $\{X_n\}$ is a sequence of variables with $X_n(w) \longrightarrow X(w)$ for all $w \in \Omega$ then*
(a) (Monotone Convergence) If $X_n(w) \geq 0$. $X_n(w) \leq X_{N+1}(w)$for all n and w, then

$$\mathbb{E}(X_n) \longrightarrow \mathbb{E}(X)$$

(b) (Dominated Convergence) If $|X_n(w)| \leq Y(w)$, $\forall n, w$.

$$\mathbb{E}(|Y|) < \infty, \text{ then } \mathbb{E}(X_n) \longrightarrow \mathbb{E}(X)$$

(c) (Bounded Convergence) If $|X_n(w)| \leq C$, $\forall C, n, w$. Then

$$\mathbb{E}(X_n) \longrightarrow \mathbb{E}(X)$$

Corollary.

$$\mathbb{E}(|X|^\alpha) \geq [\mathbb{E}(|X|)]^\alpha$$

Theorem. *Minkowoski inequality:*

$$||X + Y||_p \leq ||X||_p + ||Y||_p$$
$$||\lambda X||_p \leq |\lambda| ||X||_p$$

Definition. *X is called essentially bounded if exists $B \leq \infty$ and $\mathbb{P}\{w : |X(w)| \leq B\} = 1$*

$$||X||_\infty = \textit{inf such } B = \textit{inf}\{B \mid \mathbb{P}\{X(w) \leq B\} = 1\} = \textit{essup}\{x\}$$

$$L^\infty(\Omega) \subset \ldots L^p(\Omega) \subset \cdots \subset L^2(\Omega) \subset L^1(\Omega)$$

5 Ancillary Result

Theorem. *(Markov Inequality)*
$X \geq 0$, $\mathbb{E}(X) = \mu < \infty$

$$\mathbb{P}(X > t) \leq \frac{\mathbb{E}(X)}{t}$$

Theorem. *(chebyshev's inequality) $\mathbb{E}(\mathbb{X})^{\nvDash} < \infty$ then*

$$\mathbb{P}\left[|X - \mu| > t\right] \leq \frac{Var(X)}{t^2}$$

Theorem. *(Borel Cantelli)*

(1)If $A_1, A_2, A_3, \ldots$ are events, $\sum_{i=1}^{\infty} \mathbb{P}(A_i) < \infty$, then

$$\mathbb{P}(\textit{limsup } A_i) = \mathbb{P}(A_i i.o.) = 0$$

(2) If $A_1, A_2 \ldots$ are independent events, $\sum_{i=1}^{\infty} \mathbb{P}(A_i) = \infty$, then

$$\mathbb{P}(\textit{limsup } A_i) = 1$$

.

Definition. *Let $X_1, X_2, \ldots$ be a sequence of random variables, and let $\mathcal{F}_n$ be the σ algebra generated by $X_n, X_{n+1}, \ldots$*

$$\mathcal{F}_1 \supset \mathcal{F}_2 \supset \mathcal{F}_3 \supset \ldots$$

Define $\mathcal{F}_\infty = \cap_{i=1}^{\infty} \mathcal{F}_i$ as the tail σ algebra. If $A \in \mathcal{F}_\infty$, call A a tail event or terminal event.

Example.

$$A = \{\lim X_i \text{ exists}\} \in \mathcal{F}_\infty$$

is a tail event.

$$B = \{w : \sum_{i \to \infty} X_i(w) \text{ convergent}\}$$

is a tail event.

$$C = \{\sum_{n=1}^{\infty} X_n > 0\}$$

is NOT a tail event.

Theorem. *(Kolmogorov 0-1 theorem) Let $X_1, \dots, X_n$ be a sequence of independent random variables. Then $\forall A \in \mathcal{F}_\infty$, we have $\mathbb{P}(A) \in \{0, 1\}$*

6 Laws of Large Numbers

Theorem. *(Weak Law of Large Numbers)*
$X_1, \dots, Xn$ are independent identically distributed random variables.
$\mathbb{E}(X_i) = \mu$. Define $S_n = X_1 + \dots + X_n$ Then

$$\mathbb{P}(\{w : |\frac{S_n}{n} - \mu| > \epsilon\}) \to 0$$

Theorem. *(Strong Law of Large Numbers) $X_1, \dots, Xn$ are independent identically distributed random variables. $\mathbb{E}(X_i) = \mu < \infty$. Define $S_n = X_1 + \dots + X_n$ then*

$$\mathbb{P}(\{w : \frac{S_n(w)}{n} \to \mu\}) = 1$$

or

$$\frac{S_n(w)}{n} \to \mu \text{ a.s}$$

7 Modes of Convergence

Definition.

$$Z_n \xrightarrow{a.s.} Z \text{ if and only if } \mathbb{P}(\lim_{n \to \infty} Z_n = Z) = 1$$

$$Z_n \xrightarrow{\mathbb{P}} Z \text{ if and only if } \lim_{n \to \infty} \mathbb{P}(|Z_n - Z| \geq \epsilon) = 0$$

$$Z_n \xrightarrow{r} Z \text{ if and only if } \mathbb{E}|Z_n^r| < \infty, \mathbb{E}\left[|Z_n - Z|^r\right] = 0 \text{ as } n \to \infty$$

$$Z_n \xrightarrow{d} Z \text{ if and only if } \mathbb{P}(Z_n \leq z) \Rightarrow \mathbb{P}(Z \leq z) \text{ as } n \to \infty$$

$$\begin{matrix} (X_n \xrightarrow{\text{a.s.}} X) \searrow \\ (X_n \xrightarrow{r} X) \nearrow \end{matrix} \quad (X_n \xrightarrow{\text{P}} X) \Rightarrow (X_n \xrightarrow{\text{D}} X)$$

for any $r \geq 1$. Also, if $r > s \geq 1$ then

$$(X_n \xrightarrow{r} X) \Rightarrow (X_n \xrightarrow{s} X).$$

No other implications hold in general†.

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Theorem.
(1) If $X_n \xrightarrow{d} c$, where c is constant, then $X_n \xrightarrow{\mathbb{P}} c$.
(2) If $X_n \xrightarrow{\mathbb{P}} X$ and $\mathbb{P}(|X_n| \leq k) = 1$ for all n and some k, then $X_n \xrightarrow{r} X$ for all $r \geq 1$.
(3) If $P_n(\epsilon) = \mathbb{P}(|X_n - X| > \epsilon)$ satisfies $\sum_n P_n(\epsilon) < \infty$ for all $\epsilon > 0$, then $X_n \xrightarrow{a.s.} X$

8 Markov Chain

Definition. *S finite or countable. $X_1, X_2, \dots \in S$. Markov Property:*

$$\mathbb{P}\left[X_{n+1} = j_{n+1} | X_n = j_n, \dots, X_1 = j_1\right] = \mathbb{P}\left[X_{n+1} = j_{n+1} | X_n = j_n\right]$$

Definition.

$$f_{ij}^n = \mathbb{P}\left[X_n = j | X_{n+1} \neq j, \dots, X_1 \neq j, X_0 = i\right]$$

is the probability that starting from i, we will reach j in n steps for the first time.

$$f_{ij} = \sum_{n=1}^{\infty} f_{ij}^n$$

is the probability that starting from i, we will ever hit j.

Definition. *A state $i \in S$ is called recurrent (or persistent) if $f_i i = 1$, i.e., if starting from i, we will certainly return to i. A state $i \in S$ is called transient if it is not recurrent.*

Theorem. *Let $\{X_n\}$ be a Markov chain, on a state space S. Let $i \in S$. Then*
(1) i is transient if and only if $P(X_n = i, i.o. | X_0 = i) = 0$ if and only if $\sum_{n=i}^{\infty} p_{ii}^n < \infty$
(2) i is recurrent if and only if $P(X_n = i, i.o. | X_0 = i) = 1$ if and only if $\sum_{n=i}^{\infty} p_{ii}^n = \infty$

Definition. *A markov chain is irreducible if f_{ij} ¿ 0 , $\forall i, j \in S$, i.e., it is possible for the chain to move from any state to any other state. Equivalently, we say there exists a $r \in \mathbb{N}$, $p_{ij}^r > 0$, $\forall i, j \in S$.*

Theorem. *Let $\{p_{ij}\}_{i,j \in S}$ be the transition probabilities for an irreducible Markov chain on a state space S. Then the following are equivalent:*
1. $\exists k \in S$, $f_{kk} = 1$, i.e., k is recurrent.
2. $\forall i, j \in S$, we have $f_{ij} = 1$ (so in particular, all states are recurrent)
3. $\exists k, l$ such that $\sum_{n=1}^{\infty} p_{kl}^n = \infty$
4. $\forall i, j \in S$, $\sum_{n=1}^{\infty} p_{ij}^n = \infty$
if any (1) - (4) holds, the Markov chain itself is said to be recurrent.

Theorem. *A state S is transient, then*
(1)$\lim_{n \to \infty} P_{ij}^n = 0, \forall i, j$
(2)$\sum_{n=1}^{\infty} f_{ii}^n < \infty$

Corollary. *A state is irreducible then all states are recurrent or all states are transient.*

Definition. *Markov chain on a state S with transition probability $\{p_{ij}\}$. Let $\{\pi_i\}_{i \in S}$ be a distribution on S. $\pi_i \geq 0$, $\forall i \in S$. $\sum_{i \in S} \pi_i = 1$. We say $\{\pi_i\}_{i \in S}$ is stationary for markov chain if $\sum_{i \in S} \pi p_{ij}$, $\forall j \in S$. $[\pi]\, p = [\pi]$.*

Definition. *$\theta_i = \{u : p_{ii}^n > 0\}$ period of $i = gcd(\theta_i)$ is the probability that starting from i, we first hit j and at at time n.*

Definition. *A Markov Chain is aperiodic if the period of each state is 1.*

Lemma. *Each state in a irreducible Markov Chain has the same period.*

Definition. *A markov chain is ergodic if it is irreducible and aperiodic*

Definition. *Stationary Distribution:*

$$\pi_j > 0, \pi_j = \lim_{n \to \infty} p_{ij}^n, \pi_j = \sum_i \pi_i p_{ij}$$

Lemma. *If a Markov chain is irreducible, and has a stationary distribution $\{\pi_i\}$, then it is recurrent.*

Theorem. *If a Markov Chain is Ergodic, then:*
(1) It has a unique stationary distribution π such that $\pi p = \pi$.
(2) Regardless of the initial distribution, for all states j, $\lim_{n\to\infty} P(x_n = j) = \pi_j$.

Theorem.

$$p_{ij}^n = \sum_{k=1}^{n} f_{ij}^k p_{jj}^{n-k}$$

$$F(z) = \sum_{n=1}^{\infty} f_{ii}^n z^n$$

$$U(z) = \sum_{n=0}^{\infty} p_{ii}^n z^n$$

$$U_{ii}(z) = 1 + U_{ii}(z)F_i i(Z)$$

$$U_{ij}(z) = U_{ij}(z)F_{ij}(z)$$

Theorem. *Irreducible Markov Chain on a finite state space has a unique stationary distribution.*

9 Characteristic Function

Definition. *If X is a random variable we define the Fourier Transformation or Characteristic Function of X by:*

$$\Phi_X(t) = \hat{\mathbb{P}_X}(t) = \int e^{ity} d\mathbb{P}_x(y) = \mathbb{E}\left[e^{itx}\right]$$

Theorem. *X is random variable.*
(1). Φ_X is uniform continuous on $\mathbb{R}$
(2). $|\Phi_X| \leq 1$
(3). $Y = aX + b$. $\Phi_Y(t) = e^{itb}\Phi_X(at)$
(4). $X_1, \ldots, X_n$ are independent random variable. Then $X = \sum X_i, then \Phi_X(t) = \Pi_{i=1}^n \Phi_{X_i}(t)$
(5). If $\mathbb{E}|X|^k < \infty$, then $\Phi(t)$ is k time differentiable at 0, and $\Phi_x^k(0) = i^k \mathbb{E}\left[X^k\right]$

Theorem. *X, Y are real valued random variable. $\Phi_X = \Phi_Y$, if and only if $\mathbb{P}_x = \mathbb{P}_y$.*

Definition. *The joint characteristic function of X and Y is the function $\Phi_X, Y : \mathbb{R}^2 \longrightarrow \mathbb{R}$ given by $\Phi_{X,Y}(s,t) = \mathbb{E}(e^{isX}e^{itY})$*

Theorem. *Random Variables X and Y are independent if and only if*

$$\Phi_{X,Y}(s,t) = \Phi_X(s)\Phi_Y(t)$$

10 Central Limit Theorem

Theorem. *Let X_1, X_2, X_3, ... be i.i.d. with finite mean μ and finite non zero variance σ^2. Set $S_n = X_1 + \cdots + X_n$. Then*

$$\frac{S_n - n\mu}{\sqrt{n\sigma^2}} \xrightarrow{d} N(0,1)$$

11 Wiener Process

Definition. *The standard Wiener Process is the process* $\{W_t\}_{t\leq 0}$ *defined on* $(\Omega,\ \mathcal{F},\ \mathbb{P})$ *such that*
(1) $W_0 = 0$
(2) $W_t - W_s \sim N(0, t-s), \forall t > s$
(3) $\{W_t\}$ *has independent increments, i.e., if* $t_1 < t_2 < \cdots < t_r$*, then the random variable* $W_{t_2} - W_{t_1}$*,*$W_{t_3} - W_{t_2}$*,* $W_{t_4} - W_{t_3}$ *are independent.*
(4) W_t *is a.s continuous as a function of t, i.e.,* $\mathbb{P}\{w : t \longrightarrow W_t(w)$ *not continuous*$\} = 0$

Remark: Note the distribution of $W_t - W_s$ only depends on $t - s$.

Proposition. *1.* $\mathbb{E}(W_t) = 0$ *2.* $\mathbb{E}(W_s^2) = Var[W_s] = S$ *3.* $0 \leq S \leq t, \mathbb{E}(W_s W_t) = S$ *because* $W_s W_t = W_s(W_s + (W_t - W_s))$

12 Reference

[1] Notes from *100382 Stochastic Processes, Spring 2015, Jacobs University Bremen* taught by Dr. Keivan Mallahi Karai

[2] Probability and Random Processes, Geoffrey Grimmett and David Stirzaker, Oxford