Analysis II Recap

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1 Covering Compactness

Definition. *Let A be a subset of a metric space (X, d) and let P, Q be collections of subsets of X.*
(1) The family P is a covering of A if $A \subseteq \bigcup_{U \in P} U$
(2) The family Q is a P-subcovering of A if $Q \subseteq P$, $A \in \bigcup_{U \in Q} U$
(3) A family of sets P is called open if all $U \in P$ are open (4) The family P is finite if P consists of finitely many sets(which in turn might contain infinitely many elements of X).

Definition. *A subset A of a metric space (X, d) is called (covering) compact if every open cover P of A contains a finite subcover.*

Theorem. *Let (X, d) be a metric space, $A \subseteq X$ is a compact set. If $B \subset A$ is closed under X, then B is compact. Shortly: closed subsets of compact sets are compact.*

Theorem. *Every compact set is bounded.*

Theorem. *Any infinite subset B of a compact set A in (X,d) has at least one cluster point in A.*

Theorem. *Compact set are closed.*

Definition. *Let P be a covering of a set A in a metric space (X, d). Any number $\lambda > 0$ with the property that $\forall\ a \in A$ exist $U \in P$ such that $B_(\lambda)(a) \subseteq U$ is called a Lebesque bumber for the covering P of A.*

Theorem. *(Heine Borel) Consider the metric space $\mathbb{R}^n$ equipped with one of the standard metrics d_1, d_2 , or d_∞. Any $A \subset \mathbb{R}^n$ is compact if and only if A is closed and bounded.*

Theorem. *(X, d_X) compact, $f : (X, d_X) \longrightarrow (Y, d_Y)$ continuous. Then $f(X)$ is compact in (Y, d_Y).*

Theorem. *Any continuous function defined on compact metric spaces is uniformly continuous. That is, given a compact metric space (X, d_X), and continuous function $f : (X, d_X) \longrightarrow (Y, d_Y)$ then f is uniformly continuous.*

2 Uniform Convergence

Definition. *Let $f_n : X \longrightarrow \mathbb{R}$, $\forall n \geq 1$, is a sequence of functions.*

(1). $(f_n)_n$ converges pointwisely to $f : X \longrightarrow \mathbb{R}$ if $\forall x \in X$, $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$ such that if $n \geq \mathbb{N}$, $|f_n(x) - f(x)| < \epsilon$

(2) $(f_n)_n$ converges uniformly to $f : X \longrightarrow \mathbb{R}$ if $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$ such that $\forall x \in X$, $|f_n(x) - f(x)| < \epsilon$.

Theorem. *Suppose f_n converges uniformly to f. f_n is continuous for all n. Then f is continuous.*

3 Implicit Function Theorem

Theorem. *(Implicit Function Theorem of two variables) Let $D \subset \mathbb{R}^2$ open. F is a function in $\mathbb{R}^2$, satisfying:*

(1) $F \in C^1(D)$ ($\frac{\partial F}{\partial y}, \frac{\partial F}{\partial y}$ exists and continuous in D).

(2) $\exists (x_0, y_0) \in D$ such that $F(x_0, y_0) = 0$

(3) $\frac{\partial F}{\partial y}(x_0, y_0) \neq 0$

Then exists rectangle $I \times J \supset (x_0, y_0)$ such that:

(1): $\forall x \in I$, $\exists$ unique $y = f(x) \in J$.

(2): $f(x_0) = y_0$

(3): $f \in C^1(I)$

(4): $\forall x \in I$, $f'(x) = -\frac{\frac{\partial F}{\partial x}(x,y)}{\frac{\partial F}{\partial y}(x,y)}$, where $y = f(x)$.

Definition. *Suppose we have a system of m equations:*

$$\begin{aligned} F_1(x_1, \dots, x_n, y_1, \dots, y_n) &= 0 \\ F_2(x_1, \dots, x_n, y_1, \dots, y_n) &= 0 \\ \dots &= 0 \\ F_m(x_1, \dots, x_n, y_1, \dots, y_n) &= 0 \end{aligned} \tag{1}$$

We can write the solution of above system system as:

$$\begin{aligned} y_1 &= f_1(x_1, \dots, x_n) \\ y_2 &= f_2(x_1, \dots, x_n) \\ \dots &= \dots \\ y_m &= f_m(x_1, \dots, x_n) \end{aligned} \tag{2}$$

Can write (1) as $\vec{F}(x, y) = 0$, write (2) as $y = \vec{f}(x)$. Then $\vec{F}(x, y)$ is defined in an open set $D \subset \mathbb{R}^{n+m}$ in m $\times (n+m)$ matrix:

$$D\vec{F} = \left(\underbrace{\begin{matrix} \frac{\partial F_1}{\partial x_1} & \cdots & \frac{\partial F_1}{\partial x_n} \\ \frac{\partial F_2}{\partial x_1} & \cdots & \frac{\partial F_2}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial F_m}{\partial x_1} & \cdots & \frac{\partial F_m}{\partial x_n} \end{matrix}}_{D_x \vec{F}} \middle| \underbrace{\begin{matrix} \frac{\partial F_1}{\partial y_1} & \cdots & \frac{\partial F_1}{\partial y_m} \\ \frac{\partial F_2}{\partial y_1} & \cdots & \frac{\partial F_2}{\partial y_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial F_m}{\partial y_1} & \cdots & \frac{\partial F_m}{\partial y_m} \end{matrix}}_{D_y \vec{F}} \right)$$

$$Df = \begin{pmatrix} \frac{\partial y_1}{\partial x_1} & \cdots & \frac{\partial y_1}{\partial x_n} \\ \frac{\partial y_2}{\partial x_2} & \ddots & \frac{\partial y_2}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial y_m}{\partial x_m} & \cdots & \frac{\partial y_m}{\partial x_n} \end{pmatrix}$$

CAVEAT: Before using the Implicit Function Theorem, it is VERY important to notice which variable(s) are independent, and which are dependent. For example, In the case of finding $\frac{\partial y_i}{\partial x_i}$, y_i are dependent, x_i are independent. In the case of the problem in Midterm 2, x, y are dependent, while z is independent.

Theorem. *(Implicit Function Theorem for system of linear equations)*
Let $D \subset \mathbb{R}^{n+m}$ open. $\vec{F}$: $D \longrightarrow \mathbb{R}^m$ satisfying

(1) $\vec{F} \in C^1(D)$

(2) $\exists$ point $(x_0, y_0) \in D$ such that $F(x_0, y_0) = 0$

(3) $det D_y F(x_0, y_0) \neq 0$

Then $\exists$ $G \times H \supset (x_0, y_0)$ such that:

(1) $\forall x \in G$, $\vec{F}(x, y) = 0$ has a unique solution $f(x)$

(2) $y_0 = f(x_0)$

(3) $f \in C^1(G)$

(4) $\forall x \in G$, $Df(x) = -(D_y\vec{F}(x, y))^{-1} D_x\vec{F}(x, y)$, where $y = f(x)$.

4 Inverse Function theorem

Theorem. *(Locally Inverse Function theorem)*

$D \subset \mathbb{R}^n$ open. $f : D \longrightarrow \mathbb{R}$. Suppose

(1) $f \in C^1(D)$

(2) $\exists x_0 \in D$ such that $det Df(x_0) \neq 0$

Denote $f(x_0) = y_0$. Then exists open set $U \supset x_0$, open set $V \supset y_0$ such that:

(1) $f(U) = V$, f is injective on U

(2) Denote g as the inverse function of f on U, $g \in C^1(U)$

(3) $\forall y \in V$, $Dg(y) = (Df(x))^{-1}$, $x = g(y)$

Theorem. *(Global Inverse Function Theorem)*

$D \subset \mathbb{R}^n$ *open* $f : D \to \mathbb{R}^n$. *Suppose:*

(1) $f \in C^1(D)$

(2) $\forall x \in D$, $DetDg(x) \neq 0$ *(Then* $G = f(D)$ *open)*

(3) suppose f *is injective on* D

Then:

(1) $\exists f^{-1} : G \to D$ *such that* $\forall y \in G$, $f \cdot f^{-1}(y) = y$

(2) $f^{-1} \in C^1(G)$

(3) $Df^{-1}(y) = (Df(x))^{-1}$. *That is to say:*

$$\begin{pmatrix} \frac{\partial x_1}{\partial y_1} & \cdots & \frac{\partial x_1}{\partial y_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial x_n}{\partial y_1} & \cdots & \frac{\partial x_n}{\partial y_n} \end{pmatrix} = \begin{pmatrix} \frac{\partial y_1}{\partial x_1} & \cdots & \frac{\partial y_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial y_1}{\partial x_1} & \cdots & \frac{\partial y_n}{\partial x_n} \end{pmatrix}^{-1}$$

5 Extremas

Theorem. *Let* $U \subseteq \mathbb{R}$ *with* $\frac{\partial f}{\partial x_i}$ *exists for* $i = 1, \dots, n$. *If* $f(a)$ *is a local min or max of* a*, then* $\nabla f(a) = 0$.

Theorem. $f : U \longrightarrow \mathbb{R}$ *with* $\frac{\partial f}{\partial x_i}$ *exists for* $1, \dots, n$. *If* $f(a)$ *is a local min or max of* a*, then* $\nabla f(a) = 0$. *What is more, if (1)* Hf_a *positive definite* $\Rightarrow$ f *has a local min (2)* Hf_a *negative definite* $\Rightarrow$ f *has a local max (3)* Hf_a *indefinite* $\Rightarrow$ f *is a saddle point.*

Lemma. *The matrix*

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}$$

$\in \mathbb{R}^{n \times n}$ *is positive definite if and only if*

$$\begin{pmatrix} a_{11} & \cdots & a_{1k} \\ \vdots & \ddots & \vdots \\ a_{k1} & \cdots & a_{kk} \end{pmatrix} > 0$$

for all $k = 1, \dots, n$.

Correspondingly, A *is called negative definite if and only if*

$$\begin{pmatrix} a_{11} & \cdots & a_{1k} \\ \vdots & \ddots & \vdots \\ a_{k1} & \cdots & a_{kk} \end{pmatrix} > 0$$

for k *is even, and* < 0 *for* k *is odd.*

Corollary. *Special Case:* $n = 2$,

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

Then:
A *is positive definite if and only if* a_{11} *¿ 0, det A ¿0*

A *is negative definite if and only if* a_{11} *¡ 0, det A ¿0*

A *is indefinite if and only if det A ¡ 0*

Corollary. *Let* $(Hf)_{x_0,y_0}$ =

$$\begin{pmatrix} \frac{\partial^2 f}{\partial x^2}(x_0,y_0) & \frac{\partial^2 f}{\partial x \partial y}(x_0,y_0) \\ \frac{\partial^2 f}{\partial y \partial x}(x_0,y_0) & \frac{\partial^2 f}{\partial y^2}(x_0,y_0) \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Then if

$ac - b^2 > 0$, $a > 0$, (x_0, y_0) *is a local min*

$ac - b^2 > 0$, $a < 0$, (x_0, y_0) *is a local max*

$ac - b^2 < 0$, $(x_{0,0})$ *is a saddle point.*

Theorem. *Lagrange Multiplier.*
Suppose D *is open in* $\mathbb{R}^{m+n}$.

$$f(x_1, \ldots, x_n, y_1, \ldots, y_m) \tag{3}$$

is a function defined on D*. Now let* $x_1, \ldots, x_n$ *have constraints as follows:*

$$\begin{cases} \Phi_1(x_1, \ldots, x_n, y_1, \ldots, y_m) = 0 \\ \ldots = 0 \\ \Phi_2(x_1, \ldots, x_n, y_1, \ldots, y_m) = 0 \end{cases} \tag{4}$$

Suppose our function in (3) is under the constraint of (4), and at $z_0 = (x_0, y_0) = (a_1, \ldots, a_n, b_1, \ldots, b_m)$ *has local min or local max, then exists* $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m) \in \mathbb{R}^m$ *such that* (x_0, y_0) *is the saddle point of*

$$F(x, y) = f(x, y) + \sum_{i=1}^{m} \lambda_i \Phi_i(x, y) \tag{5}$$

That is to say, (x_0, y_0) *satisfies*

$$\begin{cases} \frac{\partial f}{\partial x_k}(x_0, y_0) + \sum_{i=1}^m \lambda_i \frac{\partial \Phi_i}{\partial x_k}(x_0, y_0) = 0, k = 1, \ldots, n \\ \frac{\partial f}{\partial y_j}(x_0, y_0) + \sum_{i=1}^m \lambda_i \frac{\partial \Phi_i}{\partial y_j}(x_0, y_0) = 0, j = 1, \ldots, m \end{cases} \tag{6}$$

6 Multivariate Calculus

Definition. *(Partial Derivatives)* $U \subset \mathbb{R}^n$ *open,* $f : U \longrightarrow \mathbb{R}$ *is partial differentiable in* j*-th direction at* $x \in U$ *if*

$$\lim_{n \to \infty} \frac{f(x + he_j) - f(x)}{h} = \frac{\partial f}{\partial x_j}(x) \tag{7}$$

Definition. $f : U \in \mathbb{R}^n \longrightarrow \mathbb{R}$

$$\textbf{\textit{grad}}\ f = \langle \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_n} \rangle \tag{8}$$

Definition. *$f : U \subset \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is called totally differentiable at a if $\exists A : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ linear, $h : U \longrightarrow \mathbb{R}^m$ such that*

$$f(a+v) = f(a) + Av + h(a+v), \forall v \in \mathbb{R}^n \tag{9}$$

such that

$$\lim_{v\to 0} \frac{h(a+v)}{||v||} = 0 \tag{10}$$

Write:

$$D_f = A = \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \dots & a_{mn} \end{pmatrix}$$

A is called the Jacobian of f at a.

Theorem. *$f : U \subset \mathbb{R}^n \longrightarrow \mathbb{R}^m$ differentiable at $a \in U$ then: (1) f is continuous at a*
(2) $f(x) = \langle f_1(x_1, \dots, x_n), \dots, f_m(x_1, \dots, x_n)\rangle$
(3) Every component of f_i is partially differentiable at a with respect to each x_j

Corollary. *f is continuously partial differentiable then f is continuous.*

Corollary. *f continuously partial differentiable $\Rightarrow$ f totally differentiable $\Rightarrow$ f partially differentiable*

Example. *(1)(Differentiable function with discontinuous partial derivatives)*

$$\begin{cases} f(x,y) = & (x^2+y^2)sin(\frac{1}{\sqrt{x^2+y^2}}), (x,y) \neq (0,0) \\ & 0, (x,y) = (0,0) \end{cases} \tag{11}$$

(2)(Partial derivatives exist but function not continuous)

$$\begin{cases} f(x,y) = & \frac{x}{x^2+y^2}, (x,y) \neq (0,0) \\ & 0, (x,y) = (0,0) \end{cases} \tag{12}$$

(3)(Partial derivatives exist and function continuous but not differentiable)

$$f(x,y) = (xy)^{1/3}$$

Theorem. *(Chain Rule) $g : U \subset \mathbb{R}^n \longrightarrow V \subset \mathbb{R}^m$*
$f : V \subset \mathbb{R}^m \longrightarrow \mathbb{R}^k$
Let g be differentiable at $x \in U$. Then $f \circ g$ is differentiable at x and

$$D_{f\circ g}(x) = D_f(g(x))D_g(x)$$

Definition. *Let $f : U \subset \mathbb{R}^n \longrightarrow \mathbb{R}$ and $x \in U$, $V \in \mathbb{R}^n$. $||v|| = 1$. The directional derivative of f at x in direction of V is*

$$D_v f(x) = \frac{d}{dt} f(x+tv) = \lim_{t\to 0} \frac{f(x+tv) - f(x)}{t}$$

Theorem. *$f : U \subset \mathbb{R}^n \longrightarrow \mathbb{R}$ continuously differentiable then $\forall x \in U$, $v \in \mathbb{R}^n$, $||v|| = 1$, have:*

$$D_v f(x) = \langle gradf(x), v\rangle = \begin{pmatrix} \frac{\partial f}{\partial x_1} & \dots & \frac{\partial f}{\partial x_n} \end{pmatrix} \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$$

7 Integration on $\mathbb{R}^d$

Will do everything on $\mathbb{R}^2$.

Definition. *Consider a rectangle(interval) $[a, b] \times [c, d] \subset \mathbb{R}^2$, $-\infty < a < b < \infty$ and $-\infty < c < d < \infty$ and partitions $P = \{a = x_0, \ldots, x_m = b\}$ and $Q = \{c = y_0, \ldots, y_n = d\}$ of $[a, b]$ and $[c, d]$. Then*

$$G = \{R_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j], i = 1, \ldots, m, j = 1, \ldots, n\}$$

is a grid of rectangles in R. For a sample set,

$$S = \{(S_{ij}, t_{ij}) \in R_{ij}, i = 1, \ldots, m, j = 1, \ldots, n\}$$

and $f : R \longrightarrow \mathbb{R}$ we define the Riemann sum

$$\mathcal{R}(f, G, S) = \sum_{i=0}^{m} \sum_{j=1}^{n} f(s_{ij}, t_{ij})|R_{ij}|$$

where $|R_{ij}| = (x_i - x_{i-1})(y_j - y_j - 1)$ is the area of the rectangle $R_{ij}, i = 1, \ldots, m, j = 1, \ldots, n$.

Definition. *A function $f : \mathbb{R} \longrightarrow \mathbb{R}$ is Riemann integrable if for some number $I \in \mathbb{R}$, such that for all $\epsilon > 0$, exists a $\delta > 0$ such that $|I - \mathcal{R}(f, G, S)| < \epsilon$ whenever mesh(G) = $max_{R_{ij} \in G} diam R_{if} < \delta$*

Definition. *The lower and upper sums of a bounded function f with respect to the grid of G are:*

$$L(f, G) = \sum_{i=1}^{m} \sum_{j=1}^{n} m_{ij}|R_{ij}| \text{ and } U(f, G) = \sum_{i=1}^{m} \sum_{j=1}^{n} M_{ij}|R_{ij}|$$

where $m_{ij} = inf f(R_{ij})$, $M_{ij} = sup f(R_{ij})$.

The lower integral of f is

$$\underline{\int} f = sup\{L(f, G), G grid in R\}$$

, and the upper integral

$$\overline{\int} inf\{U(f, G), G grid in R\}$$

Definition. *A set $Z \subset \mathbb{R}^2$ is a zeroset if for all $\epsilon > 0$, exists a countable family of open rectangles $\{S_k\}_{k \in \mathbb{N}}$ such that $Z \subseteq \cup_{k=1}^{\infty} S_k$ and $\sum_{k=1}^{\infty} |S_k| < \epsilon$.*

Theorem. *(Riemann-Lebesgue) A bounded function $f : R \longrightarrow \mathbb{R}$ if and only if the set of discontinuities of f is a zero set.*

We'll introduce Fubini theorem in a piecemeal manner.

Let $I = [a, b] \times [c, d]$, $f : I \longrightarrow \mathbb{R}$. Denote $f(x, \circ)$ as the function fixes x in $[a, b]$.$f(x, \circ *)$ is the function about the second variable.

Suppose f is bounded in $[a, b]$. Let

$$\phi(x) = \underline{\int_c^d} f(x, y)dy$$

$$\psi(x) = \overline{\int_c^d} f(x, y)dy$$

Theorem. *Suppose f is integrable in $[a,b]$. Then the single variable function ϕ and ψ is integrable on $[a,b]$. Also,*

$$\int_a^b \phi(x)dx = \int_a^b \psi(x)dx$$

That is to say,

$$\int_a^b \underline{\int_c^d} f(x,y)dydx = \int_a^b \overline{\int_c^d} f(x,y)dydx$$

Theorem. *Let f be integrable on $[a,b]\times[c,d]$. Suppose for every $x\in[a,b]$, $f(x,\circ)$ integrable in $[c,d]$. For every $y\in[c,d]$, $f(\circ,y)$ integrable in $[a,b]$, then we have*

$$\int_a^b \int_c^d f(x,y)dydx = \int_c^d \int_a^b f(x,y)dxdy$$

Corollary. *Suppose f is continuous at $[a,b]\times[c,d]$, then*

$$\int_a^b \int_c^d f(x,y)dydx = \int_c^d \int_a^b f(x,y)dxdy$$

Definition. *A bounded subset $D\subseteq\mathbb{R}^n$ is said to be a Jordan Domain if its boundary ∂D is a zeroset.*

Lemma. *Let D be a Jordan Domain and $f:D\longrightarrow\mathbb{R}$ be bounded and continuous. Let R a rectangle containing D. Then $\widetilde{f}:R\longrightarrow\mathbb{R}$ with*

$$\begin{cases}\widetilde{f}(x)= & f(x), x\in D\\ & 0, x\notin D.\end{cases} \tag{13}$$

$x\in R$ and define the Riemann integral of f on D as $\int_D f=\int_R \widetilde{f}$

Definition. *Let D be a Jordan domain. The Jordan content(or volume) vol D of D is given by vol D = $\int_D 1$*

Theorem. *(Change of Variable) Let U, $W\subseteq\mathbb{R}^2$ open and $\phi:U\longrightarrow W$ be a C^1 diffeomorphism. For $f:W\longrightarrow\mathbb{R}$ Riemann integrable on a rectangle R in U we have:*

$$\int_R f\circ\phi|DetD\phi| = \int_{\phi(R)} f$$

8 ODE

Theorem. *Solutions of First Order Linear ODE General Form: $y'+p(t)y=f(t)$*

Integrating Factor: $\mu(t)=e^{\int p(t)dt}$

General Solution: $y=\frac{1}{\mu(t)}\left(\int \mu(t)f(t)dt+C\right)$

Definition. *Let $G\in\subset\mathbb{R}\times\mathbb{R}^n$. A function $f:G\longrightarrow\mathbb{R}^k$ satisfies a Lipschitz condition(in y) with Lipschitz constant $L\geq 0$ if*

$$||f(x,y)-f(x,\widetilde{y})||\leq L||y-\widetilde{y}||$$

for all $x\in\mathbb{R}$, y, $\widetilde{y}\in\mathbb{R}$

Theorem. *Let $G\subset\mathbb{R}\times\mathbb{R}^n$ and let $f:G\longrightarrow\mathbb{R}^n$ be continuous and satisfy a local Lipschitz condition. If ϕ and ψ: $I\longrightarrow\mathbb{R}^n$, I being an interval, are 2 solutions to the system of differential equations $y'=f(x,y)$ with $\phi(x_0)=\psi(x_0)$ for some $x_0\in I$, then $\phi(x)=\psi(x)$, for all $x\in I$.*

Theorem. *Let I be an interval and* $A : I \longrightarrow \mathbb{R}^{n\times n}$, $b : I \longrightarrow \mathbb{R}^n$ *be continuous. Let* $\phi_1 = (\phi_{11}, \phi_{21}, \ldots, \phi_{n1})^T, \ldots, \phi_n = (\phi_{1n}, \phi_{2n}, \ldots, \phi_{nn})^T$ *be a fundamental system of solutions of* $y' = A(x)y$. *For,*

$$\Phi = \begin{pmatrix} \phi_{11} & \ldots & \phi_{1n} \\ \vdots & \ddots & \vdots \\ \phi_{m1} & \ldots & \phi_{mn} \end{pmatrix}$$

a solution $\psi : I \longrightarrow \mathbb{R}^n to y' = A(x)y + b(x)$ *is given by* $\psi(x) = \Phi(x)u(x)$ *with*

$$u(x) = \int_{x_0}^{x} \Phi(t)^{-1} b(t) dt + C$$

9 Reference

[1] Scripts from *100212 Analysis II, Spring 2015, Jacobs University Bremen* taught by Prof. Dr. Goetz Pfander