Applied Differential Equations Recap

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1 Classification of ODE

The **order** of DE: highest derivative appears in the equation.

ODE of order n: $F(t, y, y', y^(n))$

Linear ODE of order n:

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + ... + a_0(x) = b(x)$$

Homogeneous ODE:

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + ... + a_0(x) = 0$$

Autonomous ODE:

$$\frac{dy}{dt} = f(y)$$

2 First Order ODE

2.1 Solutions of First Order Linear ODE

General Form: $y' + p(t)y = f(t)$

Integrating Factor: $\mu(t) = e^{\int p(t)dt}$

General Solution: $y = \frac{1}{\mu(t)}\left(\int \mu(t)f(t)dt + C\right)$

2.2 Separable Equations

$M(x)dx + N(y)dy = 0 \Longrightarrow$ Method: Direct integration

2.3 Exact Equations

General Form: $M(x, y)dx + N(x, y)dy = 0$

Text for Exactness: $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$

Solution: $\phi = C$ where $M = \dfrac{\partial \phi}{\partial x}$ and $N = \dfrac{\partial \phi}{\partial y}$

Method for Solving Exact Equations:
1. Let $\phi = \int M(x, y)dx + h(y)$

2. Set $\dfrac{\partial \phi}{\partial y} = N(x, y)$

3. Simplify and solve for $h(y)$.

4. Substitute the result for $h(y)$ in the expression for ϕ from step 1 and then set $\phi = 0$. This is the solution.

2.4 Non-Exact Equations

Case 1: If $P(x, y)$ depends only on x, where

$$P(x, y) = \frac{M_y - N_x}{N} \implies \mu(y) = e^{\int P(x)dx} \tag{1}$$

then

$$\mu(x)M(x, y)dx + \mu(x)N(x, y)dy = 0 \tag{2}$$

is exact.

Case 2: If $Q(x, y)$ depends only on y, where

$$Q(x, y) = \frac{N_x - M_y}{M} \implies \mu(y) = e^{\int Q(y)dy} \tag{3}$$

Then

$$\mu(y)M(x, y)dx + \mu(y)N(x, y)dy = 0 \tag{4}$$

is exact.

2.5 Euler's Methods

$$\begin{cases} y' = f(t, y) \\ y(t_0) = y_0 \end{cases} \quad (5)$$

Can divide up the interval $[t_0, T]$ into N: h = $\frac{T-t_0}{N}$

$$y_1 = y_0 + hf(t_0, y_0)$$

....

$$y_T = y_{N-1} + hf(t_{N-1}, y_{N-1})$$

2.6 Modified Euler's Method

$$y_{N+1} = y_N + h\left[\frac{f(x_N, y_N) + f(x_N + h, y_N + hf(x_N, y_N))}{2}\right]$$

2.7 Equilibria

$x' = f(x)$. Equilibria point x_0: $f(x_0) = 0$
Suppose $f'(x_0) < 0 \Rightarrow x_0$ is asymptotically stable
Suppose $f'(x_0) > 0 \Rightarrow x_0$ is unstable

3 Second Order ODE

3.1 General Form

$y'' = f(t, y, y')$ Linear: $y'' + p(t)y' + q(t)y = g(t)$
Wronskian:

$$W(t) = y_1(t)y_2'(t) - y_1'(t)y_2(t)$$

Abel's Formula:

$$W(t) = Ce^{-\int p(t)dt}$$

3.2 Homogeneous Second Order ODE

Homogeneous:

$$ay'' + by' + cy = 0$$

Characteristic Equation:

$$a\lambda^2 + b\lambda + c = 0$$

$$\begin{cases} \text{Real Roots} : (\lambda_1 \neq \lambda_2) : y = C_1e^{\lambda_1 t} + C_2e^{\lambda_2 t} \\ \text{Repeated} : (\lambda_1 = \lambda_2) : y = (C_1 + C_2t)e^{\lambda_1 t} \\ \text{Complex} : (\lambda = \alpha \pm i\beta) : y = e^{\alpha t}(C_1 \cos \beta t + C_2 \sin \beta t) \end{cases} \quad (6)$$

reduction of order

Suppose we have y_1, then we can find y_2 by

$$y_2 = y_1 \int \frac{e^{-\int p(x)dx}dx}{y_1(x)^2} \quad (7)$$

4 Non Homogeneous ODE

4.1 Strategy

1. Find all of the solutions of corresponding homogeneous ODE, $y(t)$

2. Find a particular solution of the non homogeneous ODE, $y_2(t)$

3. General solution to the non homogeneous ODE is: $y(t) + y_2(t)$

4.2

Method of Judicious Guessing (Ansatz)

$g(t) =$	$y_2(t) =$
$Ce^{\alpha t}$	$Ae^{\alpha t}t^s$
$C_1Sin\beta t + C_2Cos\beta t$	$t^s(ASin\beta t + BCos\beta t)$
$C_n(t^n) + C_{n-1}(t^{n-1}) + ... + C_0$	$t^s(A_0 + A_1t + \cdots + A_nt^n)$

4.3 Cauchy Euler Equation

ODE $ax^2y'' + bxy' + cy = 0$

Characteristics Function: $a\lambda^2 + (b - a)\lambda + c = 0$

The solutions of depend on the roots $\lambda_{1,2}$

$$\begin{cases} \text{Real Roots} : y = C_1x^{\lambda_1} + C_2x^{\lambda_2} \\ \text{Repeated} : y = C_1x^{\lambda} + C_2x^{\lambda}ln(x) \\ \text{Complex} : y = x^{\alpha}[C_1 \cos(\beta \ln x) + C_2 \sin(\beta \ln x)] \end{cases} \quad (8)$$

$\lambda_{1,2} = \alpha \pm i\beta$, where α,$\beta \in \mathbb{R}$

5 Systems of two first order ODEs

5.1 General Form

$$X' = \begin{pmatrix} x_1'(t) \\ x_2'(t) \end{pmatrix} = \underbrace{\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}}_{A} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} \quad (9)$$

5.2 Solutions

If A has:

$$\begin{cases} \textbf{two distinct Eigenvalues}: \\ X(t) = C_1 e^{\lambda_1 t} V_1 + C_2 e^{\lambda_2 t} V_2 \\ \textbf{a pair of Complex eigenvalues}: \\ X(t) = C_1 e^{\alpha}(Cos\beta ta - Sin\beta tb) + C_2 e^{\alpha}(Sin\beta ta + Cos\beta tb), \\ \text{ where } \lambda_{1,2} = \alpha \pm \beta i, v_{1,2} = a \pm bi \\ \textbf{one eigenvalue}: X(t) = C_1 e^{\lambda t} v + C_2 e^{\lambda t}(tv + w), \\ \text{ where } (A - \lambda I)w = v \end{cases} \tag{10}$$

5.3 Equilibria

$$\begin{cases} x_1' = f(x, y) \\ x_2' = g(x, y) \end{cases} \tag{11}$$

The system attains a equilibria when:

$$\begin{cases} f(x, y) = 0 \\ g(x, y) = 0 \end{cases} \tag{12}$$

For a particular $\tilde{x}$

$$\begin{cases} f'(\tilde{x}) > 0 : \text{the equilibria is unstable} \\ f'(\tilde{x}) < 0 : \text{the equilibria is asymtotically stable} \end{cases} \tag{13}$$

Stability properties of linear systems $\mathbf{x}' = \mathbf{A}\mathbf{x}$ with $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$ and $\det \mathbf{A} \neq 0$.

Eigenvalues	Type of Critical Point	Stability
$\lambda_1 > \lambda_2 > 0$	Node	Unstable
$\lambda_1 < \lambda_2 < 0$	Node	Asymptotically stable
$\lambda_2 < 0 < \lambda_1$	Saddle point	Unstable
$\lambda_1 = \lambda_2 > 0$	Proper or improper node	Unstable
$\lambda_1 = \lambda_2 < 0$	Proper or improper node	Asymptotically stable
$\lambda_1, \lambda_2 = \mu \pm i\nu$	Spiral point	
$\mu > 0$		Unstable
$\mu < 0$		Asymptotically stable
$\lambda_1 = i\nu, \lambda_2 = -i\nu$	Center	Stable

6 System of first order ODE in general

6.1 General Form

$$X' = \begin{pmatrix} x_1'(t) \\ \vdots \\ x_n'(t) \end{pmatrix} = \underbrace{\begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \vdots & \vdots \\ a_{n1} & \dots & a_{nn} \end{pmatrix}}_{A} \begin{pmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{pmatrix} \tag{14}$$

6.2 Solution

$X(t) = e^{tA} X_0$ is the solution to the initial value problem

$$\begin{cases} X' = AX \\ X(0) = X_0 \end{cases} \tag{15}$$

7 Laplace Transformation

7.1 General Formula

$F(s) = \mathcal{L}\{f\} = \int_0^\infty f(t) e^{-st} dt$ c is a real number.

7.2 Solution Table

	$f(t) = \mathcal{L}^{-1}\{F(s)\}$	$F(s) = \mathcal{L}\{f(t)\}$
1.	1	$\frac{1}{s}, \quad s > 0$
2.	e^{at}	$\frac{1}{s-a}, \quad s > a$
3.	$t^n, \quad n = \text{positive integer}$	$\frac{n!}{s^{n+1}}, \quad s > 0$
4.	$t^p, \quad p > -1$	$\frac{\Gamma(p+1)}{s^{p+1}}, \quad s > 0$
5.	$\sin at$	$\frac{a}{s^2 + a^2}, \quad s > 0$
6.	$\cos at$	$\frac{s}{s^2 + a^2}, \quad s > 0$
7.	$\sinh at$	$\frac{a}{s^2 - a^2}, \quad s > \lvert a \rvert$
8.	$\cosh at$	$\frac{s}{s^2 - a^2}, \quad s > \lvert a \rvert$
9.	$e^{at} \sin bt$	$\frac{b}{(s-a)^2 + b^2}, \quad s > a$
10.	$e^{at} \cos bt$	$\frac{s-a}{(s-a)^2 + b^2}, \quad s > a$
11.	$t^n e^{at}, \quad n = \text{positive integer}$	$\frac{n!}{(s-a)^{n+1}}, \quad s > a$
12.	$u_c(t)$	$\frac{e^{-cs}}{s}, \quad s > 0$
13.	$u_c(t) f(t-c)$	$e^{-cs} F(s)$
14.	$e^{ct} f(t)$	$F(s-c)$
15.	$\int_0^t f(t-\tau) g(\tau)\, d\tau$	$F(s)G(s)$
16.	$\delta(t-c)$	e^{-cs}
17.	$f^{(n)}(t)$	$s^n F(s) - s^{n-1} f(0) - \cdots - f^{(n-1)}(0)$
18.	$t^n f(t)$	$(-1)^n F^{(n)}(s)$

7.3 Properties

Linearity

$\mathcal{L}\{c_1 f_1 + c_2 f_2\} = c_1 \mathcal{L}\{f_1\} + c_2 \mathcal{L}\{f_2\}$

Scaling

Assume $\mathcal{L}\{f(t)\} = F(s)$
$\Longrightarrow \mathcal{L}\{f(at)\} = \frac{1}{a} F(\frac{s}{a})$

Shift Property 1

Assume $\mathcal{L}\{f(t)\} = F(s)$, C is a real number.
$\Longrightarrow \mathcal{L}\{e^{ct} f(t)\} = F(s - C)$

Shift Property 2

$\mathcal{L}\{f(t-c)u_c(t)\} = F(s)e^{-cs}$

7.4 Derivatives

$$\begin{cases} \mathcal{L}\{f(t)\} = F(s) \\ \mathcal{L}\{f'(t)\} = sF(s) - f(0) \\ \mathcal{L}\{f''(t)\} = s^2 F(s) - sf(0) - f'(0) \\ .... \\ \mathcal{L}\{f^n(t)\} = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - f^{n-1}(0) \end{cases} \tag{16}$$

7.5 Step Function

$$u_c(t) = \begin{cases} 0, t < c \\ 1, t \geq c \end{cases} \tag{17}$$

Suppose $f(t)$ is a piecewise function.

$$f(t) = \begin{cases} f_1(t), t \leq c \\ f_2(t), c_1 \leq t \leq c_2 \\ f_3(t), c_2 \leq t \leq c_3 \\ \ldots \\ f_n(t), c_{n-1} \leq t \end{cases} \tag{18}$$

We can write $f(t)$ as

$$f(t) = f_1(t) + u_{c_1}(t)\left[f_2(t) - f_1(t)\right] + u_{c_2}(t)\left[f_3(t) - f_2(t)\right] + \ldots u_{c_n-1}(t)\left[f_n(t) - f_{n-1}(t)\right] \tag{19}$$

7.6 Dirac Function

$$\int_0^\infty f(t)\,\delta(t) dt = f(0) \tag{20}$$

7.7 Periodic

Suppose f is periodic, i.e., $f(x+T) = f(x)$ for some T

$$\mathcal{L}\{f(t)\} = \frac{1}{1 - e^{-sT}} \int_0^T f(t) e^{-st} dt \tag{21}$$

8 Solving DE with Laplace Transformation

1. Do Laplace Transform on both sides of DE and plug in the initial value

2. Compute $Y(s)$

3. $y(t) = \mathcal{L}^{-1}\{Y(s)\}$ by using partial fraction decomposition

9 Reference

[1] Notes from *110262 Applied Differential Equations and Modeling, Spring 2015, Jacobs University Bremen* taught by Dr. Keivan Mallahi Karai

[2] Differential Equations: An Introduction to Modern Methods and Applications. James R. Brannan, William E. Boyce