URYSOHN'S THEOREM AND TIETZE EXTENSION THEOREM

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Definition 0.1. Let $x, y \in$ topological space X. We define the following properties of topological space X:

T_0: If $x \neq y$, there is an open set containing x but not y or an open set containing y but not x.

T_1: If $x \neq y$, there is an open set containing y but not x.

T_2: If $x \neq y$, there are disjoint open sets U, V with $x \in U$ and $y \in V$.

T_3: X is a T_1 space, and for any closed set $A \subset X$ and any $x \in A^C$ there are disjoint open sets U, V with $x \in U$ and $A \subset V$.

T_4: X is a T_1 space, and for any disjoint closed sets A, B in X there are disjoint open sets U, V with $A \subset U$and $B \subset V$.

We say a T_2 space is a Hausdorff space, a T_3 space is a regular space, a T_4 space is a normal space.

Definition 0.2. $C(X, [a, b]) \coloneqq$ Space of all continuous $[a, b]$ valued functions on X.

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Theorem 0.3. (Urysohn's Lemma) Let X be a normal space. If A and B are disjoint closed sets in X, there exists $f \in C(X, [0,1])$ such that $f = 0$ on A and $f = 1$ on B.

Proof.

Step 1: *Define a large collection of open sets in* X (Lemma 4.14 in [1])
Let D be the set of dyadic rationals in $[0,1]$, that is, $D = \{1, 0, \frac{1}{2}, \frac{1}{4}, \frac{3}{4}, \frac{1}{8}, \frac{3}{8}, \frac{7}{8} \dots\}$.

We want to construct a sequence of open sets U_q in X, indexed by $q \in D$, such that:
(1) For each $q \in D$, $A \subseteqq U_q$
(2) $B \subset U_1$ and for each $q < 1$, $B \cap U_q = \varnothing$
(3) $\forall p, q \in D$ with $p < q$, we have $\overline{U_p} \subseteqq U_q$

To this end, we define
$U_1 = X \backslash B$
$\overline{U_0} \subseteqq U_1$ is closed $\Rightarrow \exists$ open $U_0 \supseteqq A$ such that $\overline{U_0} \subseteqq U_1$
$\overline{U_0} \subseteqq U_{\frac{1}{2}}$ is closed $\Rightarrow \exists$ open $U_{\frac{1}{4}} \supseteqq \overline{U_0}$ such that $\overline{U_{\frac{1}{4}}} \subseteqq U_{\frac{1}{2}}$
$\overline{U_{\frac{1}{2}}} \subseteqq U_1$ is closed $\Rightarrow \exists$ open $U_{\frac{3}{4}} \supseteqq \overline{U_{\frac{1}{2}}}$ such that $\overline{U_{\frac{3}{4}}} \subseteqq U_1$
...
This construction satisfies (1), (2), and (3).
Step 2: *Define the map* f

Define:

$$f : X \to [0,1]$$

via

$$x \mapsto \inf\{q | x \in U_q\}$$

By (1), $f = 0$ in A. By (2), $f = 1$ in B.

Step 3: *Prove the map* f *is continuous*
It is sufficient to show that $f^{-1}(-\infty, \alpha)$and $f^{-1}(\alpha, \infty)$ are open because we know half lines generate the topology on $\mathbb{R}$.

$f(x) < \alpha \iff x \in U_r$ for some $r < \alpha \iff x \in \bigcup_{r<\alpha} U_r$

$\Rightarrow f^{-1}(-\infty, \alpha) = \bigcup_{r<\alpha} U_r$ is open.

$f(x) > \alpha \iff x \notin U_r$ for some $r > \alpha \iff x \notin \overline{U_s}$ for some $s > \alpha \iff x \in \bigcup_{s>\alpha} (\overline{U_s})^C$

$\Rightarrow f^{-1}(\alpha, \infty) = \bigcup_{s>\alpha} (\overline{U_s})^C$ is open.

$\square$

Theorem 0.4. (Tietze Extension Theorem) Let X be a normal space and A be a closed subset in X. If $f : A \to [a,b]$ is a continuous function, then f has a continuous extension $F : X \to [a,b]$, i.e., F is continuous and $F|_A = f$.

Proof. WLOG, assume that $[a,b] = [0,1]$. (just replace f by $\frac{f-a}{b-a}$.

Step 1.
Let $h : A \to [0,k]$ be a continuous function for some constant k. Then $B = h^{-1}([0,\frac{k}{3}])$ and $C = h^{-1}([\frac{2k}{3},k])$ are closed in A, and therefore closed in X. By Urysohn Lemma, $\exists g : X \to [0,\frac{k}{3}]$ such that $g(B) = 0$ and $g(C) = \frac{k}{3}$.
Note that $g \le \frac{k}{3}$, and $h - g \le \frac{2k}{3}$ on A.

Step 2.
Start with $f : A \to [0,1]$. (k=1 in Step 1.)
Then $g_1 \le 1/3$.
$f - g_1 : A \to [0, 2/3]$.
Now apply Step 1 to this function $f - g_1$ with $k = 2/3$:
Then $f - g_1 - g_2 \le (2/3)^2$ and $g_2 \le 1/3 \cdot 2/3 = 2/9$.
In this way, we can obtain a sequence $\{g_n\}$ with the property that
(1)$f - g_1 - g_2 \cdots - g_n \le (2/3)^n$
(2)$g_n \le 1/3 \cdot (2/3)^{n-1}$.

Step 3.
Let $s_n = g_1 + g_2 \cdots + g_n$.
Then (s_n) is a Cauchy sequence in $\mathcal{C}(X,\mathbb{R})$ since
$\|s_n - s_m\| = \|g_{n+1} + \cdots + g_m\| \le \|g_{n+1}\| + \cdots + \|g_m\|$
$\le (1/3)((2/3)^n + \cdots + (2/3)^{m-1}) < (1/3)(2/3)^n(1+2/3+(2/3)^2+\cdots) = (2/3)^n$
By the completeness of $\mathcal{C}(X,\mathbb{R})$, $s_n \to F$ uniformly and $F \in \mathcal{C}(X,\mathbb{R})$.
Now we claim that F is a desired extension of f:
Step 2(1) $\Rightarrow \|f - s_n\|_A \le (2/3)^n \Rightarrow s_n \to f$ uniformly on A$\Rightarrow F = f$ on A.

$\square$

References

[1]Folland *Real Analysis*