Presentation on Kalman Filter

Tianlin Liu

March 23 2015

1 Recap

1.1 How do we describe the states of system?

$x_t = F_t x_{t-1} + B_t u_t + W_t$
x_t: State vector
F_t: State transition matrix
B_t: Control input matrix
W_t: Process noise

1.1.1 Example

$x_t = x_{t-1} + \dot{x}_{t-1}\Delta t + \frac{f_t(\Delta t)^2}{2m}$

$$\begin{bmatrix} x_t \\ \dot{x}_t \end{bmatrix} = \begin{bmatrix} 1 & \Delta t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_{t-1} \\ \dot{x}_{t-1} \end{bmatrix} + \begin{bmatrix} \frac{(\Delta t)^2}{2} \\ \Delta t \end{bmatrix} \frac{f_t}{m}$$

1.2 How do we describe the measurement of the system?

$$z_t = H_t x_t + v_t$$

z_t: measurement of the state vector.
H_t: transformation matrix that connects state vector and measurement vector.
V_t: measurement noise.

1.3 How do we describe the stages of the algorithm?

1.3.1 Predict

$$\hat{x}_{t|t-1} = F_t \hat{x}_{t-1|t-1}$$

$$P_{t|t-1} = F_t P_{t-1|t-1} F_t^T + Q_t$$

1.3.2 Update

$$\hat{x}_{t|t} = \hat{x}_{t|t-1} + K_t(Z_t - H_t \hat{x}_{t|t-1}), \text{ where} K_t = P_{t|t-1} H_t^T (H_t P_{t|t-1} H_t^T + R_t)^{-1}$$

$$P_{t|t} = (I - K_t H_t) P_{t|t-1}$$

These four formulas in the predict and update steps are most important part for studying Kalman Filter. In last session, we only gave out these four formulas, but we did not prove them rigorously. Now, let's do this in this session.

2 Let's determine K_t

$$z_t = H_t x_t + v_t$$
$$z_t - H_t \hat{x}_{t|t-1} = H_t x_t + v_t - H_t \hat{x}_{t|t-1}$$
$$\Rightarrow \hat{x}_{t|t} = \hat{x}_{t|t-1} + K_t(H_t x_t + v_t - H_t \hat{x}_{t|t-1})$$

Recall the definition of the covariance matrix,

$$P_{t|t} = E[(x_t - \hat{x}_{t|t})(x_t - \hat{x}_{t|t})^T] = E(e_t e_t^T)$$

What is $x_t - \hat{x}_{t|t}$?

$$x_t - \hat{x}_{t|t} = x_t - (\hat{x}_{t|t-1} + K_t(H_t x_t + v_t - H_t \hat{x}_{t|t-1}))$$
$$= x_t - \hat{x}_{t|t-1} - K_t H_t x_t - K_t v_t + K_t H_t \hat{x}_{t|t-1}$$
$$= x_t - \hat{x}_{t|t-1} - K_t H_t x_t + K_t H_t \hat{x}_{t|t-1} - K_t v_t$$
$$= (I - K_t H_t)(x_t - \hat{x}_{t|t-1}) - K_t v_t$$
$$\Rightarrow P_{t|t} = E[(\underbrace{(I - K_t H_t)}_{A}\underbrace{(x_t - \hat{x}_{t|t-1})}_{e'_t} - K_t v_t)(\underbrace{(I - K_t H_t)}_{A}\underbrace{(x_t - \hat{x}_{t|t-1})}_{e'_t} - K_t v_t)^T]$$
$$= E[(Ae'_t - K_t v_t)(Ae'_t - K_t v_t)^T]$$
$$= E[(Ae'_t - K_t v_t)(e'^T_t A^T - v_t^T K_t^T)]$$
$$= E[Ae'_t e'^T_t A^T - Ae'_t v_t^T K_t^T - K_t v_t e'^T_t A^T + K_t v_t v_t^T K_t^T]$$
$$= AE[e'_t e'^T_t]A^T - \underbrace{AE[e'_t v_t^T]K_t^T}_{0} - \underbrace{K_t E[v_t e'^T_t]A^T}_{0} + K_t E[v_t v_t^T]K_t^T$$

Since the state estimate error e'_t and process noise term $(K_t)^T$ are uncorrelated.

$$= AE[e'_t e'^T_t]A^T + K_t E[v_t v_t^T]K_t^T$$
$$= (I - K_t H_t)\underbrace{E[(x_t - \hat{x}_{t|t-1})(x_t - \hat{x}_{t|t-1})^T]}_{P_{t|t-1}}(I - K_t H_t)^T + K_t \underbrace{E[v_t v_t^T]}_{R} K_t^T$$
$$= (I - K_t H_t)P_{t|t-1}(I - K_t H_t)^T + K_t R K_t^T$$
$$= (P_{t|t-1} - K_t H_t P_{t|t-1})(I - K_t H_t)^T + K_t R K_t^T$$
$$= (P_{t|t-1} - K_t H_t P_{t|t-1})(I - H_t^T K_t^T) + K_t R K_t^T$$
$$= P_{t|t-1} - P_{t|t-1} H_t^T K_t^T - K_t H_t P_{t|t-1} + K_t H_t P_{t|t-1} H_t^T K_t^T + K_t R K_t^T$$
$$= P_{t|t-1} - P_{t|t-1} H_t^T K_t^T - K_t H_t P_{t|t-1} + K_t(H_t P_{t|t-1} H_t^T K_t^T + R)K_t^T$$
$$\Rightarrow Tr(P_{t|t}) = Tr(P_{t|t-1}) - 2Tr(P_{t|t-1} H_t^T K_t^T) + Tr(K_t(H_t P_{t|t-1} H_t^T K_t^T + R)K_t^T)$$

Fact: $\frac{\delta Tr(A)}{\delta B} = 0$, $\frac{\delta Tr(A)}{\delta A} = I$, $\frac{\delta Tr(AB)}{\delta A} = B^T$, $\frac{\delta Tr(BA^T)}{\delta A} = B$, $\frac{\delta Tr(ABA^T)}{\delta A} = 2AB$

To minimize the variances: $\frac{\delta Tr(P_{t|t})}{\delta K_t} = 0 = -2P_{t|t-1}H^T + 2K_t(H_t P_{t|t-1} H_t^T K_t^T + R)$

$$\Rightarrow K_t(H_t P_{t|t-1} H_t^T + R) = P_{t|t-1} H^T$$
$$\Rightarrow K_t = P_{t|t-1} H_t^T (H_t P_{t|t-1} H_t^T + R)^{-1}$$

3 Let's determine $P_{t+1|t}$

Define $H_t P_{t|t-1} H_t^T + R = S_t$

$$\Rightarrow K_t = P_{t|t-1} H_t^T (H_t P_{t|t-1} H_t^T + R)^{-1} = P_{t|t-1} H_t^T (S_t)^{-1}$$

Plug K_t into

$$P_{t|t} = P_{t|t-1} - P_{t|t-1} H_t^T K_t^T - K_t H_t P_{t|t-1} + K_t (H_t P_{t|t-1} H_t^T K_t^T + R) K_t^T$$

Note that $H_t P_{t|t-1} H_t^T K_t^T + R = \frac{P_{t|t-1} H_t^T}{K_t}$ So,

$$P_{t|t} = P_{t|t-1} - P_{t|t-1} H_t^T K_t^T - K_t H_t P_{t|t-1} + \underbrace{K_t (\frac{P_{t|t-1} H_t^T}{K_t}) K_t^T}_{P_{t|t-1} H_t^T K_t^T}$$

$$= P_{t|t-1} - K_t H_t P_{t|t-1} = (I - K_t H_t)(P_{t|t-1})$$

Recall F is the state transition matrix: $\hat{x}_{t+1|t} = F_t \hat{x}_{t|t}$.
Then,

$$e_{t+1|t} = x_{t+1} - \hat{x}_{t+1|t+1} = (F_{t+1} x_t + w_t) - F_{t+1} x_{t|t}$$
$$= F_t e_t + w_t$$

$$P_{t+1|t} = E[e_{t+1|t} e_{t+1|t}^T] = E[(F_{t+1} e_t + w_t)(F_{t+1} e_k + w_t)^T]$$

Since e_k and w_k have zero cross- correlation,

$$P_{t+1|t} = E[e_{t+1|t} e_{t+1|t}^T] = E[(F_{t+1} e_{t|t} (F_{t+1} e_{t|t})^T] + \underbrace{E[w_t w_t^T]}_{Q}$$

$$= F_{t+1} P_{t|t} F_{t+1}^T + Q$$

$$\Rightarrow P_{t|t-1} = F_t P_{t-1|t-1} F_t^T + Q$$

as desired.

4 References

[1] *Tutorial: The Kalman Filter*, Tony Lacey,
http://www0.cs.ucl.ac.uk/staff/s.prince/4C75/LaceyThackerTutorial.pdf
[2] *Understanding the Basis of the Kalman Filter Via a Simple and Intuitive Derivation*, Ramsey Faragher, IEEE Signal Processing Magazine Sept. 2012