BAIRE CATEGORY THEOREM AND OPEN MAPPING THEOREM

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1. The Baire Category Theorem

Definition 1.1. Let X be a topological space. $A \subset X$.

- If $\overline{A} = X$, then A is called a dense set.
- If $(\overline{A})^\circ = \varnothing$, then A is called nowhere dense set.

Remark 1.2. A set $A \subset X$ is dense if and only if there exists some points of A in every non-empty open set of X.

Theorem 1.3. (The Baire Category Theorem) Let X be a complete metric space.

(1) If $\{U_n\}_{n=1}^\infty$ is a sequence of open dense subsets of X, then $\bigcap_{n=1}^\infty U_n$ is dense in X.
(2) X is not a countable union of nowhere dense sets.

Proof. (1) Pick any $W \in X$, W open and non-empty. By Remark 1.2, it is enough to show that $W \cap (\bigcap_{n=1}^\infty U_n) \neq \varnothing$.
Idea: We want to find a Cauchy sequence $\{x_i\}$ such that

$$x_i \in (W_n \cap \bigcap_{n=1}^{i} U_n)$$

Then we can use the Completeness property to argue that the limit of $\{x_i\}$ lies in $W \cap (\bigcap_{n=1}^\infty U_n)$ and therefore it is non-empty.

We construct this sequence of $\{x_i\}$, in the following way:

Since U_1 is a set of open and dense subset of X, W is open and non empty, $U_1 \cap W$ is open and non empty(by Remark 1.2). Hence there exists a ball $B(r_1, x_1)$, which is centered at x_1 and with radius r_1 such that $B(r_1, x_1) \subset (U_1 \cap W)$. Assume $0 < r_1 < 1$.

Then inductively, we choose x_n, and r_n. We observe that $U_n \cap B(r_{n-1}, x_{n-1})$ is open and non empty, so we can choose x_n, r_n such that $0 < r_n < 2^{-n}$ and $\overline{B(r_n, x_n)} \subset U_n \cap B(r_{n-1}, x_{n-1})$. Then for $n, m \geq N$, we have $x_n, x_m \in B(r_N, x_N)$. Since $r_n \to 0$, $\{x_n\}$ is

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a Cauchy sequence. Since X is complete, $x = \lim_{n\to\infty} x_n$ exists. Since $x_n \in B(r_N, x_N)$, $\forall n \geq N$, we have

$$x \in \overline{B(r_N, x_N)} \subset U_N \cap B(r_1, x_1) \subset U_N \cap W$$

for all N. Hence we prove the claim.

(2) Let $\{E_n\}$ be a sequence of nowhere dense set. Then $\{(\overline{E_n})^C\}$ is a sequence of open dense sets. By part (1) we know that

$$\bigcap (\overline{E_n})^C \neq \varnothing$$

Hence

$$(\bigcup \overline{E_n})^C = \bigcap (\overline{E_n})^C \neq \varnothing \Rightarrow \bigcup \overline{E_n} \neq X$$

That implies

$$\bigcup E_n \subset \bigcup \overline{E_n} \neq X$$

□

Definition 1.4. Let X be a topological space.
(1) We say $E \subset X$ is of the first category, or say E is a meager set if E is a countable union of nowhere dense set.
(2) If $E \subset X$ is not of the first category, we say E is of the second category, or E is a residual set.

2. Open Mappting Theorem

Definition 2.1. Let X, Y be topological Space. A map $f : X \to Y$ is called open if $f(U)$ is open in Y whenever U is open in X.

Definition 2.2. Let $B_r = \{x \in X, \|x\| < r\}$ denote the open ball of radius $r > 0$ centered at $0 \in X$, and let $B_r(x_0)$ be a ball centered at x_0.

Theorem 2.3. Let X, Y be Banach spaces. Let $B(X, Y)$ denote the space of all bounded linear maps from X to Y. If $T \in B(X, Y)$ is surjective, then it is an open mapping.

Proof. It is enough to prove that the image $T(B_1)$ of the open unit ball centered at $0 \in X$ contains an open ball around $0 \in Y$.

Claim 2.4. (Step 1) $\overline{T(B_{\frac{1}{2}})}$ has non empty interior, i.e., there is some ball $B_\epsilon(y_0)$ centered at some point y_0, $B_\epsilon(y_0) \subset \overline{T(B_{\frac{1}{2}})}$

Proof. Clearly,

$$X = \bigcup_{n=1}^{\infty} B_{\frac{n}{2}}$$

Since T is surjective,

$$Y = \bigcup_{n=1}^{\infty} T(B_{\frac{n}{2}})$$

Y is Banach and therefore complete. The Baire Category Theorem now guarantees that there must be some $n_0 \in \mathbb{N}$ such that $\overline{T(B_{n_0/2})}$ has non-empty interior. This implies that

$$n_0 \overline{T(B_{1/2})} = \overline{T(B_{n_0/2})}$$

has non empty interior. Hence $\overline{T(B_{1/2})}$ has non empty interior. It follows that there is some ball $B_\epsilon(y_0)$ centered at some point y_0, $B_\epsilon(y_0) \subset \overline{T(B_{\frac{1}{2}})}$. □

Claim 2.5. (Step 2) $B_\epsilon(y_0) - y_0 = B_\epsilon \subset \overline{T(B_1)}$

Proof. By the previous claim, it is of course enough to show that $T(B_{\frac{1}{2}}) - y_0 \subset \overline{T(B_1)}$. So let $y \in \overline{T(B_{\frac{1}{2}})}$ and find a sequence $\{y_n\} \subset T(B_{\frac{1}{2}})$ such that $y_n \to y$, and a sequence $\{z_n\} \subset T(B_{\frac{1}{2}})$ such that $z_n \to y_0$. Let $\{x_n\}, \{w_n\} \subset B_{\frac{1}{2}}$ be such that $T(x_n) = y_n$ and $T(w_n) = z_n$. Then we have that $y_n - z_n \to y - y_0$. Also notice that $\|x_n - w_n\| < 1$, so that $\{x_n - w_n\}$ is a sequence in B_1, which implies that $y - y_0 \in \overline{T(B_1)}$, proving the claim. □

Claim 2.6. (Step 3) $\overline{T(B_{1/2^n})}$ contains $B_{\epsilon/2^n}$.

Proof. By linearity:

$$\overline{T(B_{1/2^n})} = \overline{1/2^n T(B_1)} = 1/2^n \overline{T(B_1)}$$

where $1/2^n \overline{T(B_1)}$ by the previous claim contains $(1/2^n)B_\epsilon = B_{\epsilon/2^n}$. □

Claim 2.7. (Step 4, almost done!) $B_{\frac{\epsilon}{2}}$ is contained in $T(B_1)$.

Proof. Let $y \in B^Y_{\frac{\epsilon}{2}}$. We will show that $y \in T(B^X_1)$. From the previous claim, we know that $y \in \overline{T(B_{\frac{1}{2}})}$. This means that we can find some $y_1 \in T(B_{\frac{1}{2}})$ approximating y by $\|y - y_1\| < \frac{\epsilon}{4}$. But $y_1 \in T(B_{\frac{1}{2}})$ implies that there is some $x_1 \in B^X_{1/2}$ that maps to y_1, so we have that $\|y - T(x_1)\| < \frac{\epsilon}{4}$. Therefore, $y - T(x_1) \in B^Y_{\frac{\epsilon}{4}}$. Following the same reasoning, we find $x_2 \in B^X_{\frac{1}{4}}$ such that $\|y - T(x_1) - T(x_2)\| < \frac{1}{2}$. Proceeding inductively, we produce elements $x_n \in B^X_{1/2^n}$ such that

$$\|y - \sum_{k=1}^{n} T(x_k)\| < \frac{\epsilon}{2^{n+1}} \tag{2.1}$$

Now let $z_n = x_1 + \dots + x_n$. Notice that for $m < n$,

$$\|z_n - z_m\| = \|x_{m+1} + \dots x_n\|_1 \leq \sum_{k=m+1}^{n} \|x_k\| \leq \sum_{k=m+1}^{n} \frac{1}{2^k} = \frac{1}{2^m} \to 0$$

Hence $\{z_n\}$ is a Cauchy Sequence, and therefore convergent sequence in X. $z_n \to x$ for some $x \in X$.

Further,

$$\|x\| = \lim_{n\to\infty} \|z_n\| = \lim_{n\to\infty} \|x_1 + \dots x_n\| \leq \lim_{n\to\infty} \sum_{k=1}^{n} \|x_k\| < \frac{1}{2} + \lim_{n\to\infty} \sum_{k=2}^{n} \|x_k\| = \frac{1}{2} + \frac{1}{2} = 1$$

Hence $x \in B^X_1$. By continuity of T and (2.1), $T(x) = y$. Hence $y \in T(B_1)$

□

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References

[1]Folland *Real Analysis*